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SOME HOMOLOGICAL PROPERTIES OF THE CATEGORY $\mathcal{O}$, II

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ABSTRACT. We show, in full generality, that Lusztig's $\mathbf{a}$-function describes the projective dimension of both indecomposable tilting modules and indecomposable injective modules in the regular block of the BGG category $\mathcal{O}$, proving a conjecture from the first paper. On the way we show that the images of simple modules under projective functors can be represented in the derived category by linear complexes of tilting modules. These complexes, in turn, can be interpreted as the images of simple modules under projective functors in the Koszul dual of the category $\mathcal{O}$. Finally, we describe the dominant projective module and also projective-injective modules in some subcategories of $\mathcal{O}$ and show how one can use categorification to decompose the regular representation of the Weyl group into a direct sum of cell modules, extending the results known for the symmetric group (type A).

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1. Introduction

Let $\mathfrak{g}$ be a semi-simple complex finite-dimensional Lie algebra and $\mathcal{O}_0$ the principal block of the BGG category $\mathcal{O}$ for $\mathfrak{g}$ ([BGG]). After the Kazhdan-Lusztig conjecture, formulated in [KL] and proved in [BB, BK], it became clear that many algebraic properties of $\mathcal{O}_0$ can be studied using the Kazhdan-Lusztig combinatorics ([KL, BjBr]). In the first paper [Ma2] I conjectured that the projective dimension of an indecomposable tilting modules in $\mathcal{O}_0$ is given by Lusztig's $\mathbf{a}$-function ([Lu1, Lu2]). In [Ma2] this conjecture was proved in the case $\mathfrak{g} = \mathfrak{sl}_n$ (type A). The proof consisted of two parts. In the first part it was shown that the projective dimension of an indecomposable tilting module in $\mathcal{O}_0$ is an invariant of a two-sided cell (this part does not depend on the type of $\mathfrak{g}$). The second part was computational, computing the projective dimension for certain indecomposable tilting modules, however, the computation was based on the Koszul self-duality of $\mathcal{O}_0$ ([So1]) and computations of graded filtrations of certain modules in $\mathcal{O}_0$ in type A ([Ir2]). Computations in [Ir2] and in the subsequent paper

[IS] covered also some special cases for other types and to these case the arguments from [Ma2] extend naturally. However, from [IS] it was also known that the arguments from [Ir2] and [IS] certainly do not extend to the general case. Hence, to connect tilting modules and Lusztig's **a**-function in full generality one had to come up with a completely different approach then the one I proposed in [Ma2].

The main objective of the present paper is to prove the mentioned above conjecture from [Ma2] in full generality. We also prove a similar conjecture from [Ma2] about the projective dimension of indecomposable injective modules. The proposed argument makes a surprising connection to another part of the paper [Ma2]. The category $\mathcal{O}_0$ is equivalent to the category of modules over a finite-dimensional Koszul algebra ([BGG, So1]), in particular, one can consider the corresponding category $\mathcal{O}_0^{\mathbb{Z}}$ of graded modules. In this situation an important role is played by the category of the so-called linear complexes of tilting modules ([Ma1, MO2, MOS]). A part of [Ma2] is dedicated to showing that many structural modules from $\mathcal{O}_0$ (and from the parabolic subcategories of $\mathcal{O}_0$ in the sense of [RC]) can be described using linear complexes of tilting modules. In the present paper we establish yet another class of such modules, namely, the modules obtained from simple modules using projective functors ([BG]). In fact, we even show that with respect to the Koszul self-duality of $\mathcal{O}_0$ this class of modules is Koszul self-dual (other families of Koszul self-dual modules, for example shuffled Verma modules, can be found in [Ma2]). After this we show that certain numerical invariants of those linear complexes of tilting modules, which represent the images of simple modules under projective functors, are given in terms of Lusztig's **a**-function. The conjecture from [Ma2] follows then using computations in the derived category.

By [BG], the action of projective functors on $\mathcal{O}_0$ can be considered as a categorification of the right regular representation of the Weyl group of $\mathfrak{g}$ (or the corresponding Hecke algebra in the case of the category $\mathcal{O}_0^{\mathbb{Z}}$, see [MS4]). The images of simple modules under certain projective functors appear in [MS3] for the case $\mathfrak{g} = \mathfrak{sl}_n$ as categorical interpretations of elements in a certain basis, in which the regular representation of the symmetric group decomposes into a direct sum of cell modules (which are irreducible in type A). In the present paper images of simple modules under projective functors appear naturally in the general case. So, we extend the above result to the general case, generalizing [MS3], which, in particular, establishes certain interesting facts about these modules. For example, we show that the images of simple modules under projective functors, which appear in our picture, have simple head and simple socle. We also confirm [KM, Conjecture 2] about the structure of the dominant projective module in certain subcategories of $\mathcal{O}_0$.

The paper is organized as follows: In Section 2 we describe the setup and introduce all necessary notation. In Section 3 we study the images of simple modules under projective functors, in particular, we establish their Koszul self-duality and extend the results from [MS3] to the general case. In Section 4 we prove the conjecture from [Ma2] about the connection between the projective dimension of indecomposable tilting modules in $\mathcal{O}_0$ and Lusztig's **a**-function. In Section 5 we prove the conjecture from [Ma2] about the connection between the projective dimension of indecomposable injective modules in $\mathcal{O}_0$ and Lusztig's **a**-function.

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2. Preliminaries

2.1. **Category $\mathcal{O}$.** I refer the reader to [BGG, Hu, Ma2, MS4] for more details on the category $\mathcal{O}$ and the notation I will use. Let W denote the Weyl group of $\mathfrak{g}$ and w_0 be the longest element of W. Denote by $\mathtt{A}$ the basic finite-dimensional associative algebra, whose category $\mathtt{A}$-mod of (left) finite-dimensional modules is equivalent to $\mathcal{O}_0$ ([BGG]). We fix the Koszul grading $\mathtt{A} = \bigoplus_{i\geq 0} \mathtt{A}_i$ on $\mathtt{A}$ ([So1, BGS]) and denote by $\mathtt{A}$-gmod the category of finite-dimensional graded $\mathtt{A}$-modules with degree zero morphisms (this category is equivalent to the category $\mathcal{O}_0^{\mathbb{Z}}$ mentioned above). For $k \in \mathbb{Z}$ we denote by $\langle k\rangle$ the autoequivalence of $\mathtt{A}$-gmod, which shifts (decreases) the degree of homogeneous components of a graded module by k.

Simple modules in $\mathcal{O}_0$ are indexed by elements of W in the natural way. For $w \in W$ we denote by $L(w)$ the simple graded $\mathtt{A}$-module corresponding to w, concentrated in degree zero (here $L(e)$ corresponds to the trivial module in $\mathcal{O}_0$ and $L(w_0)$ corresponds to the simple Verma module in $\mathcal{O}_0$). We denote by $P(w)$ the projective cover of $L(w)$ in $\mathtt{A}$-gmod; by $I(w)$ the injective envelop of $L(w)$ in $\mathtt{A}$-gmod; by $\Delta(w)$ the standard quotient of $P(w)$ (a Verma module) and by $\nabla(w)$ the costandard submodule of $I(w)$ (a dual Verma module). Finally, we denote by $T(w)$ the indecomposable tilting module, corresponding to w, whose grading is uniquely determined by the condition that $\Delta(w)$ is a submodule of $T(w)$.

For $w \in W$ we denote by θ_w the graded version ([St]) of the indecomposable projective endofunctor of $\mathtt{A}$-gmod corresponding to w ([BG]). The functor θ_w is normalized by the condition $\theta_w P(e) \cong P(w)$ (as graded modules). The functor θ_w is both left and right adjoint to $\theta_{w^{-1}}$.

Denote by $D^b(\mathtt{A})$ the bounded derived category of $\mathtt{A}$-gmod and by $[\![k]\!]$, $k \in \mathbb{Z}$, the autoequivalence of $D^b(\mathtt{A})$, which shifts complexes by k positions to the left. We denote by $\mathfrak{LT}$ the full subcategory of $D^b(\mathtt{A})$, which consists of all complexes

$$\mathcal{X}^\bullet : \qquad \dots \to \mathcal{X}^{-1} \to \mathcal{X}^0 \to \mathcal{X}^1 \to \dots$$

such that for every $i \in \mathbb{Z}$ the module $\mathcal{X}^i$ is isomorphic to a direct sum of modules of the form $T(w)\langle i\rangle$, $w \in W$. The category $\mathfrak{LT}$ is abelian with enough projectives, moreover, there is an equivalence of categories $\Phi : \mathfrak{LT} \to \mathtt{A}$-gmod ([MO2, Ma1]). From [Ma1, Theorem 3.3(1)] we have that Φ sends the indecomposable tilting module $T(w)$, $w \in W$, considered as a linear complex concentrated in position zero, to the simple module $L(w_0w^{-1}w_0)$.

We denote by $\star : D^b(\mathtt{A}) \to D^b(\mathtt{A})$ the usual contravariant autoequivalence preserving isoclasses of simple modules concentrated in degree zero (duality). All projective functors commute with $\star$. For $M \in \mathtt{A}$-gmod, $M \neq 0$, we set

$$\min(M) = \min\{i \in \mathbb{Z} : M_i \neq 0\}, \qquad \max(M) = \max\{i \in \mathbb{Z} : M_i \neq 0\}.$$

For $w \in W$ we denote by $\mathtt{T}_w : \mathtt{A}\text{-gmod} \to \mathtt{A}\text{-gmod}$ the corresponding Arkhipov's twisting functor (see [AS, KhMa] for the ungraded version and [MO2, Appendix] for the graded version).

2.2. **Kazhdan-Lusztig combinatorics.** Here I refer the reader to [MS4, Section 3], [So2] and [BjBr] for details. Let S be the set of simple reflections in W and $l : W \to \mathbb{Z}$ be the length function with respect to S. Denote by $\mathbb{H}$ the Hecke algebra of W, which is a free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{H_w : w \in W\}$ and multiplication given by

$$H_xH_y = H_{xy} \text{ if } l(x) + l(y) = l(xy); \text{ and } H_s^2 = H_e + (v^{-1} - v)H_s, s \in S.$$

Let $\{\underline{H}_w : w \in W\}$ and $\{\underline{\hat{H}}_w : w \in W\}$ denote the Kazhdan-Lusztig and the dual Kazhdan-Lusztig bases of $\mathbb{H}$, respectively.

Consider the Grothendieck group $[\mathtt{A}\text{-gmod}]$ of $\mathtt{A}$-gmod and for $M \in \mathtt{A}$-gmod denote by $[M]$ the image of M in $[\mathtt{A}\text{-gmod}]$. Then the assignment $[\Delta(w)\langle i\rangle] \mapsto v^{-i}H_w$ gives rise to an isomorphism between $[\mathtt{A}\text{-gmod}]$ and $\mathbb{H}$. In what follows we will often identify $[\mathtt{A}\text{-gmod}]$ and $\mathbb{H}$ via this isomorphism. For all $M \in \mathtt{A}$-gmod we have $[M\langle i\rangle] = v^{-i}[M]$. We also have $[P(w)] = \underline{H}_w$ and $[L(w)] = \underline{\hat{H}}_w$ for all $w \in W$. Furthermore, for any $w \in W$ and $M \in \mathtt{A}$-gmod we have $[\theta_w M] = [M]\underline{H}_w$. All the above extends to $D^b(\mathtt{A})$ in the obvious way.

Further, we denote by $\leq_L$, $\leq_R$ and $\leq_{LR}$ the left, the right and the two-sided orders on W, respectively (to make things coherent with [MS4] our convention is that e is the minimal element and w_0 is the maximal element). The equivalence classes with respect to these orders are called left-, right- and two-sided cells of W, respectively. The corresponding equivalence relations will be denoted by $\sim_L$, $\sim_R$ and $\sim_{LR}$,

respectively. Let $\mathbf{a} : W \to \mathbb{Z}$ be Lusztig's $\mathbf{a}$-function on W ([Lu1, Lu2]). This function respects the two-sided order, in particular, it is constant on two-sided cells. On the (unique) distinguished (Duflo) involution w from a given left cell we have $\mathbf{a}(w) = l(w) - 2\delta(w)$, where $\delta(w)$ is the degree of the Kazhdan-Lusztig polynomial $P_{e,w}$. Since $\mathbf{a}$ is constant on two-sided cells, we sometimes will write $\mathbf{a}(X)$, where X is cell (left, right, or two-sided), meaning $\mathbf{a}(x)$, $x \in X$.

Any right cell $\mathbf{R}$ comes equipped with a natural dual Kazhdan-Lusztig basis. Multiplication with elements from the Kazhdan-Lusztig basis respects the right preorder and produces elements, which are linear combinations of elements from the dual Kazhdan-Lusztig basis of $\mathbf{R}$ or smaller right cells. Taking the quotient defines on the linear span of elements from the dual Kazhdan-Lusztig basis of $\mathbf{R}$ the structure of an $\mathbb{H}$-module, called the (right) cell module.

2.3. **Subcategories of $\mathcal{O}$ associated with right cells.** For a fixed right cell $\mathbf{R}$ of W let $\hat{\mathbf{R}}$ denote the set of all elements $x \in W$ such that $x \leq_R w$ for some $w \in \mathbf{R}$. Let $\{e_w : w \in W\}$ be a complete set of primitive idempotents of $\mathtt{A}$ corresponding to our indexing of simple modules. Following [MS4] set $e(\hat{\mathbf{R}}) = \sum_{w \notin \hat{\mathbf{R}}} e_w$ and consider the quotient

$$\mathtt{A}^{\hat{\mathbf{R}}} = \mathtt{A}/\mathtt{A}e(\hat{\mathbf{R}})\mathtt{A}.$$

Then $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod is a subcategory of $\mathtt{A}$-gmod in the natural way (this is the Serre subcategory of $\mathtt{A}$-gmod generated by the simple modules of the form $L(w)\langle k\rangle$, $w \in \hat{\mathbf{R}}$, $k \in \mathbb{Z}$). The category $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod is stable under all projective functors θ_w, $w \in W$. See [MS4] for details. The structural modules in $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod will be normally denoted similarly to the corresponding modules from $\mathtt{A}$-gmod but with an extra index $\hat{\mathbf{R}}$.

We denote by $\mathrm{Z}^{\hat{\mathbf{R}}} : \mathtt{A}\text{-gmod} \to \mathtt{A}^{\hat{\mathbf{R}}}\text{-gmod}$ the left adjoint of the natural inclusion of $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod into $\mathtt{A}$-gmod. The functor $\mathrm{Z}^{\hat{\mathbf{R}}}$ is just the functor of taking the maximal possible quotient which belongs to $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod. This functor commutes with all projective functors [MS4, Lemma 19] and satisfies

$$\mathrm{Z}^{\hat{\mathbf{R}}}L(w) = \begin{cases} L^{\hat{\mathbf{R}}}(w), & w \in \hat{\mathbf{R}}; \\ 0, & \text{otherwise}; \end{cases} \qquad \mathrm{Z}^{\hat{\mathbf{R}}}P(w) = \begin{cases} P^{\hat{\mathbf{R}}}(w), & w \in \hat{\mathbf{R}}; \\ 0, & \text{otherwise}. \end{cases}$$

In the case when $\mathbf{R}$ contains an element of the form xw_0, where x is the longest element in some parabolic subgroup of W, the category $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod is equivalent to the graded version of the parabolic category $\mathcal{O}$ in the sense of [RC]. In this case $\mathrm{Z}^{\hat{\mathbf{R}}}$ is the corresponding Zuckerman functor ([MS2]).

3. Images of simple modules under projective functors

In this section we will study modules $M(x,y) = \theta_x L(y)$, $x, y \in W$. On the level of the Hecke algebra we have $[M(x,y)] = \underline{\hat{H}}_y \underline{H}_x \in \mathbb{H}$. The latter elements of $\mathbb{H}$ play an important role in the combinatorics of $\mathbb{H}$, see [Lu1, Lu2, Mat, Ne].

3.1. **Graded lengths of $M(x,y)$.** We start with the following result which describes nonzero homogeneous components of the module $M(x,y)$.

Proposition 1. *For all $x, y \in W$ we have:*
(a) $M(x,y)^\star \cong M(x,y)$;
(b) $M(x,y) \neq 0$ if and only if $x \leq_R y^{-1}$.
(c) $\max(M(x,y)) = -\min(M(x,y)) = \mathbf{a}(y)$ whenever $x \sim_R y^{-1}$, and $\max(M(x,y)) = -\min(M(x,y)) < \mathbf{a}(y)$ whenever $x <_R y^{-1}$.

Proof. The module $L(y)$ is simple and hence selfdual (i.e. satisfies $L(y)^\star \cong L(y)$). Now the claim (a) follows from the fact that $\star$ and θ_x commute. This implies the equality $\max(M(x,y)) = -\min(M(x,y))$ in (c).

The rest of the statements is purely combinatorial and follows from well-known properties of (dual) Kazhdan-Lusztig bases in the Hecke algebra. To start with, the claim (b) follows from [Mat, (1.4)].

That $\max(M(x,y)) = \mathbf{a}(y)$ in the case $x \sim_R y^{-1}$ follows from the definition of $\mathbf{a}$ and the explicit formula for $\underline{\hat{H}}_y \underline{H}_x$, see for example [Mat, Lemma 2.1]. Similarly, that $\max(M(x,y)) < \mathbf{a}(y)$ whenever $M(x,y) \neq 0$ and $x <_R y^{-1}$ follows from the definition of $\mathbf{a}$, [Mat, Lemma 2.1] and the fact that $\mathbf{a}$ respects the right order ([Lu1, Corollary 6.3]). This completes the proof. $\square$

3.2. **The dominant projective module in $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod.** Our next goal is to show that modules $M(x,y)$ (for certain choices of x and y) are projective-injective modules in the category $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod. To prove this we first have to describe the dominant projective module $P^{\hat{\mathbf{R}}}(e)$ in $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod. In what follows we assume that $\mathbf{R}$ is a fixed right cell of W.

Proposition 2. *Let $x \in \mathbf{R}$. There is a unique (up to scalar) nonzero homomorphism from $\Delta(e)$ to $\theta_{x^{-1}} L(x)$ and the module $P^{\hat{\mathbf{R}}}(e)$ coincides with the image of this homomorphism.*

Proof. By adjunction we have

$$\begin{aligned} \mathrm{Hom}_{\mathtt{A}}(\Delta(e), \theta_{x^{-1}} L(x)) &= \mathrm{Hom}_{\mathtt{A}}(\theta_x \Delta(e), L(x)) \\ &= \mathrm{Hom}_{\mathtt{A}}(P(x), L(x)) \\ &= \mathbb{C}, \end{aligned}$$

which proves the first part of the claim. Let D denote the image of $\Delta(e)$ in $\theta_{x^{-1}} L(x)$. By [Ka, Proposition 5.1], the module D does not depend

on the choice of x. Choose x the maximal possible (with respect to the Bruhat order). Then the only simple subquotient from $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod in the module $T(x)$ is $L(x)$, so we have the complex

$$0 \to M_{-1} \to T(x) \to M_1 \to 0,$$

which has only one nonzero homology, namely $L(x)$ in the zero position. All simple subquotients of both M_{-1} and M_1 do not belong to $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod. Applying the exact functor $\theta_{x^{-1}}$ we get the complex

$$0 \to \theta_{x^{-1}} M_{-1} \to \theta_{x^{-1}} T(x) \to \theta_{x^{-1}} M_1 \to 0, \tag{1}$$

which has only one nonzero homology, namely $\theta_{x^{-1}} L(x)$ in the zero position. Note that this homology belongs to $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod as $L(x)$ does and $\theta_{x^{-1}}$ preserves this category.

Since $T(x)$ has a unique occurrence of $L(x)$ (counting all shifts in grading as well), by adjunction there is a unique (up to scalar) nonzero morphism from $\Delta(e)$ to $\theta_{x^{-1}} T(x)$, which is a tilting module. Since $\Delta(e)$ is the dominant Verma module, we get that $\theta_{x^{-1}} T(x)$ must contain $T(e)$ as a direct summand (with multiplicity one, counting with all shifts) and the above homomorphism from $\Delta(e)$ to $\theta_{x^{-1}} T(x)$ is the natural injection from $\Delta(e)$ into this direct summand.

Let $y \in W$. Then, by adjunction, for every $i \in \mathbb{Z}$ we have

$$\mathrm{Hom}_{\mathtt{A}}(\theta_{x^{-1}} M_{-1}, L(y)\langle i \rangle) = \mathrm{Hom}_{\mathtt{A}}(M_{-1}, \theta_x L(y)\langle i \rangle).$$

Since θ_x preserves $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod, the space on the right hand side can be nonzero only if $y \not\in \hat{\mathbf{R}}$. It thus follows that every simple module occurring in the head of M_{-1} does not belong to $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod. Similarly, all simple modules occurring in the socle of $\theta_{x^{-1}} M_1$ do not belong to $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod.

From the above we know that the module $\theta_{x^{-1}} T(x)$ has a unique simple subquotient isomorphic to $L(e)$ (counting all shifts of grading), which, moreover, appears in the homology of the sequence (1). This means that the monomorphism from $\Delta(e)$ to $\theta_{x^{-1}} T(x)$ induces a homomorphism from $P^{\hat{\mathbf{R}}}(e) = \mathrm{Z}^{\hat{\mathbf{R}}} \Delta(e)$ to the homology $\theta_{x^{-1}} L(x)$. From the two previous paragraphs it follows that this homomorphism is injective. This completes the proof. □

Corollary 3. *Let d be the Duflo involution in $\mathbf{R}$. The module $P^{\hat{\mathbf{R}}}(e)$ has simple socle $L(d)\langle -\mathbf{a}(d) \rangle$, and all other composition subquotients of the form $L(x)\langle -i \rangle$, where $x <_{LR} d$ and $0 \leq i < \mathbf{a}(d)$.*

Proof. This follows from Proposition 2 and [Ka, Proposition 5.1]. □

Remark 4. Proposition 2 proves [KM, Conjecture 2].

Remark 5. Using [Ka, Proposition 5.1] one can relate the module $P^{\hat{\mathbf{R}}}(e)$ to a primitive quotient of the universal enveloping algebra $U(\mathfrak{g})$.

3.3. **Projective-injective modules in $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod.** Using the results from the previous subsection we obtain the following:

Theorem 6. *Let $d \in \mathbf{R}$ be the Duflo involution. Then the modules $M(x,d)$, $x \in \mathbf{R}$, are exactly the indecomposable projective-injective modules in $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod (up to shift).*

Proof. Let $x \in \mathbf{R}$. Applying θ_x to the short exact sequence

$$0 \to L(d)\langle -\mathbf{a}(d)\rangle \to P^{\hat{\mathbf{R}}}(e) \to \mathrm{Coker} \to 0,$$

given by Corollary 3, we obtain the short exact sequence

$$0 \to M(x,d)\langle -\mathbf{a}(d)\rangle \to P^{\hat{\mathbf{R}}}(x) \to \theta_x\mathrm{Coker} \to 0.$$

From Corollary 3 and Proposition 1(b) we have $\theta_x\mathrm{Coker} = 0$ and hence $M(x,d)\langle -\mathbf{a}(d)\rangle \cong P^{\hat{\mathbf{R}}}(x)$. This shows that the module $M(x,d)$ is projective in $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod. From Proposition 1(a) we have that $M(x,d)$ is self-dual, hence it is injective in $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod as well.

On the other hand, for $x \in \hat{\mathbf{R}} \setminus \mathbf{R}$ we have that $P^{\hat{\mathbf{R}}}(x) = \theta_x P^{\hat{\mathbf{R}}}(e)$ has simple top $L(x)$. At the same time, as $P^{\hat{\mathbf{R}}}(e)$ has simple socle $L(d)$ (up to shift), using adjunction and arguments similar to those used in the proof of Proposition 2, one shows that every simple submodule of $P^{\hat{\mathbf{R}}}(x)$ must have the form $L(y)$, $y \in \mathbf{R}$ (up to shift). Therefore the injective cover of $P^{\hat{\mathbf{R}}}(x)$ does not coincide with $P^{\hat{\mathbf{R}}}(x)$ by the previous paragraph. Hence the module $P^{\hat{\mathbf{R}}}(x)$ is not injective. This completes the proof. □

The following result generalizes some results of [Ir1]:

Corollary 7. *The Loewy length of every projective-injective module in $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod equals $2\mathbf{a}(\mathbf{R})+1$.*

Proof. Let X be an indecomposable projective-injective module in the category $\mathtt{A}^{\hat{\mathbf{R}}}$-gmod. From Theorem 6 and Proposition 1 we obtain that $2\mathbf{a}(\mathbf{R})+1$ is the graded length of this module (the number of nonzero homogeneous components). As the algebra $\mathtt{A}$ is Koszul, it is positively graded and generated in degrees zero and one. Hence the quotient algebra $\mathtt{A}^{\hat{\mathbf{R}}}$ is positively graded and generated in degrees zero and one as well. Since X has both simple socle and simple head (by Theorem 6), from [BGS, Proposition 2.4.1] we thus obtain that the graded filtration of X is a Loewy filtration. The claim follows. □

Corollary 8. *The injective envelope of $P^{\hat{\mathbf{R}}}(e)$ is $P^{\hat{\mathbf{R}}}(d)\langle \mathbf{a}(d)\rangle$.*

Proof. This follows from Theorem 6 and Corollaries 3 and 7. □

Corollary 9. *Let $d \in \mathbf{R}$ be the Duflo involution. Then all modules $M(x,d)$, $x \in \mathbf{R}$, have their simple socles and simple heads in degrees $\mathbf{a}(x)$ and $-\mathbf{a}(x)$, respectively.*

Proof. This follows from the positivity of the grading on $\mathtt{A}$, Proposition 1 and Theorem 6. □

Remark 10. Projective-injective modules play important role in the structure and properties of the category $\mathcal{O}$ and related categories, see [Ir1, MS1, MS4] and references therein.

3.4. **Application to Kostant's problem.** Results from the previous subsections can be applied to one classical problem in Lie theory, called Kostant's problem ([Jo]). If M, N are two $\mathfrak{g}$-modules, then $\mathrm{Hom}_{\mathbb{C}}(M, N)$ is a $\mathfrak{g}$-bimodule in the natural way. Denote by $\mathcal{L}(M, N)$ the subbimodule of $\mathrm{Hom}_{\mathbb{C}}(M, N)$, which consists of all elements, the adjoint action of $\mathfrak{g}$ on which is locally finite. Then for any $\mathfrak{g}$-module M the universal enveloping algebra $U(\mathfrak{g})$ maps naturally to $\mathcal{L}(M, M)$ inducing an injection

$$U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M) \hookrightarrow \mathcal{L}(M, M). \tag{2}$$

Kostant's problem for M is to determine whether the latter map is surjective. The problem is very hard and the answer is not even known for the modules $L(w)$, $w \in W$, in the general case, although many special cases are settled (see [Jo, Ma3, MS3, MS4, Ka, KM] are references therein). Taking into account the results of the previous subsections, the main result of [KM] can be formulated as follows:

Theorem 11 ([KM]). *Let $d \in \mathbf{R}$ be the Duflo involution. Then Kostant's problem for $L(d)$ has a positive answer (i.e. the map from* (2) *is surjective) if and only if the only simple modules occurring in the socle of the cokernel of the natural injection $P^{\hat{\mathbf{R}}}(e) \hookrightarrow P^{\hat{\mathbf{R}}}(d)\langle \mathbf{a}(d)\rangle$ (given by Corollary 8) are (up to shit) the modules of the form $L(x)$, $x \in \mathbf{R}$.*

After Theorem 6 the above can be reformulated in terms of the so-called double-centralizer property (see [So1, KSX, MS5]). Let A be a finite-dimensional algebra and X be a left A-module. Then A has the double centralizer property with respect to X if there is an exact sequence

$$0 \to {}_A A \to X_1 \to X_2,$$

where both X_1 and X_2 are isomorphic to finite direct sums of some direct summands of X. Double centralizer properties play important role in the representation theory (see [So1, KSX, MS5]). In our case we have:

Corollary 12. *Let $d \in \mathbf{R}$ be the Duflo involution. Then Kostant's problem for $L(d)$ has a positive answer if and only if $\mathtt{A}^{\hat{\mathbf{R}}}$ has the double centralizer property with respect to the direct sum of all indecomposable projective-injective modules.*

Proof. Let X denote the cokernel of the natural inclusion $P^{\hat{\mathbf{R}}}(e) \hookrightarrow P^{\hat{\mathbf{R}}}(d)\langle \mathbf{a}(d)\rangle$ given by Corollary 8. Assume that Kostant's problem

has a positive answer for $L(d)$. Then, because of Theorem 11 and Theorem 6, the injective envelope I of X is also projective and hence we have an exact sequence

$$0 \to P^{\hat{\mathbf{R}}}(e) \to P^{\hat{\mathbf{R}}}(d)\langle \mathbf{a}(d)\rangle \to I,$$

where the two last terms are both projective and injective. Applying θ_x, $x \in \hat{\mathbf{R}}$, gives the exact sequence

$$0 \to P^{\hat{\mathbf{R}}}(x) \to \theta_x P^{\hat{\mathbf{R}}}(d)\langle -\mathbf{a}(d)\rangle \to \theta_x I,$$

where again the two last terms are both projective and injective since θ_x preserves both projectivity and injectivity. This implies that $\mathtt{A}^{\hat{\mathbf{R}}}$ has the double centralizer property with respect to the direct sum of all indecomposable projective-injective modules.

On the other hand, if Kostant's problem has a negative answer for $L(d)$, then, because of Theorem 11 and Theorem 6, the injective envelope I of X is not projective. Therefore $\mathtt{A}^{\hat{\mathbf{R}}}$ does not have the double centralizer property with respect to the direct sum of all indecomposable projective-injective modules. This completes the proof. $\square$

Remark 13. In the case $\mathfrak{g} = \mathfrak{sl}_n$ (type A) Corollary 12 controlls the answer to Kostant's problem for all $L(w)$, $w \in W$, as this answer is known to be a left cell invariant ([MS4]).

3.5. **Regular $\mathbb{H}$-module as a sum of cell modules.** In this subsection we extend the results of [MS3] and [KMS] to the general case. For $w \in W$ let d_w denote the Duflo involution in the right cell of W. For $x \in W$ denote by $[\theta_x]$ the linear operator on $[\mathtt{A}\text{-gmod}]$, induced by the exact functor θ_x. We have the following categorification result:

Theorem 14. *(a) The action of* $[\theta_x]$, $x \in W$, *on* $[\mathtt{A}\text{-gmod}]$ *gives a right regular representation of* $\mathbb{H}$ *in the Kazhdan-Lusztig basis.*

(b) The classes $[M(w, d_w)]$, $w \in W$, *form a basis of the complex vector space* $\mathbb{C} \otimes_{\mathbb{Z}} [\mathtt{A}\text{-gmod}]/(v-1)$, *on which the action of* $[\theta_x]$, $x \in W$, *gives a right regular representation of* W.

(c) In the basis from (b) *the right regular representation of* W *decomposes into a direct sum of (right) cell modules.*

Proof. Using Theorem 6 and Corollary 8 the proof is similar to that of the main result from [MS3]. $\square$

Remark 15. Theorem 14 gives a category theoretical interpretation of a basis in Lusztig's asymptotic Hecke algebra ([Lu2, Ne]).

3.6. **Koszul self-duality.** In this subsection we prove the following crucial result which establishes Koszul self-duality for modules $M(x, y)$:

Theorem 16. *Let* $x, y \in W$.

(a) There is $\mathcal{M}(x, y)^\bullet \in \mathfrak{LT}$ *that has a unique nonzero homology which is in position zero and is isomorphic to* $M(x, y)$.

(b) $\Phi\mathcal{M}(x,y)^\bullet \cong M(y^{-1}w_0, w_0x^{-1})$.

Proof. We prove both statements by a descending induction on the length of y. Let $w \in W$. Consider $T(w)$ as a linear complex concentrated in position zero (this is a simple object of $\mathfrak{LT}$). We have $\Phi T(w) \cong L(w_0w^{-1}w_0)$. On the other hand, we also have

$$T(w) \cong \theta_{w_0w}T(w_0) \cong \theta_{w_0w}\Delta(w_0) \cong \theta_{w_0w}L(w_0) = M(w_0w, w_0).$$

Hence our complex consisting of $T(w)$ is exactly $\mathcal{M}(w_0w, w_0)^\bullet$ and we have

$$\Phi\mathcal{M}(w_0w, w_0)^\bullet \cong M(e, w_0w^{-1}w_0),$$

which agrees with (b). This proves the basis of the induction.

Assume now the the statement is true for all $y \in W$ such that $l(y) > k$, where $0 \leq k < l(w_0)$, and let $y \in W$ be such that $l(y) = k$. Let $s \in S$ be such that $l(y^{-1}w_0s) < l(y^{-1}w_0)$ and $\overline{y} = w_0(y^{-1}w_0s)^{-1}$. Then $l(\overline{y}) > k$ and hence the claim of the theorem is true for all $M(x,\overline{y})$, $x \in W$, by the inductive assumption.

For $x \in W$ take the linear complex $M(x,\overline{y})^\bullet$. We have $\Phi M(x,\overline{y})^\bullet \cong M(y^{-1}w_0s, w_0x^{-1})$. As $l(y^{-1}w_0ss) > l(y^{-1}w_0s)$ by our choice of s, applying θ_s to $M(y^{-1}w_0s, w_0x^{-1})$, using [Ma2, (1)], and going back to $\mathfrak{LT}$ via Φ^{-1}, we obtain a direct sum of linear complexes, where one direct summand will be $\Phi^{-1}M(y^{-1}w_0, w_0x^{-1})$ and multiplicities of other direct summand are determined by Kazhdan-Lusztig's μ-function ([KL, BjBr]) as given by [Ma2, (1)].

The Koszul dual of θ_s is the derived Zuckerman functor ([RH, MOS]). This functor was explicitly described in [MS2]. It has only three components. The first one takes the maximal quotient with subquotients of the form $L(w)$, $l(w_0sw_0w) > l(w)$, (with the corresponding shifts in grading) and shifts it one position to the right. This component is zero because of our choice of s. The second component is dual to the first one. It takes the maximal submodule with subquotients of the form $L(w)$, $l(w_0sw_0w) > l(w)$, (with the corresponding shifts in grading) and shifts it one position to the left. This component is zero by the dual reason and Proposition 1(a). The only component which is left is the functor Q from [MS2, Theorem 5], so the homology of the complex $\Phi^{-1}\theta_sM(y^{-1}w_0s, w_0x^{-1})$ is isomorphic to $\mathrm{Q}M(x, w_0sw_0y)$ (and the homology of $\Phi^{-1}M(y^{-1}w_0, w_0x^{-1})$ is a direct summand which, as we will see, is easy to track).

Let us now compute the module $\mathrm{Q}M(x, w_0sw_0y)$. From [MS2, Theorem 5] and [KhMa] it follows that the functor Q commutes with projective functors, in particular, we have

$$\mathrm{Q}M(x, w_0sw_0y) \cong \mathrm{Q}\theta_xL(w_0sw_0y) \cong \theta_x\mathrm{Q}L(w_0sw_0y).$$

Using [MS2, Theorem 5] and [AS, Theorem 6.3] we obtain that the module $\mathrm{Q}L(w_0sw_0y)$ is a direct sum of $L(y)$ and some other simple modules, whose multiplicities are again determined by Kazhdan-Lusztig's μ-function. Hence, comparing [AS, Theorem 6.3] and [Ma2, (1)] and using the inductive assumption we see that these "other simple modules" precisely correspond to the direct summands of the linear complex $\Phi^{-1}\theta_s M(y^{-1}w_0s, w_0x^{-1})$, different from $\Phi^{-1}M(y^{-1}w_0, w_0x^{-1})$. Therefore the homology of $\Phi^{-1}M(y^{-1}w_0, w_0x^{-1})$ is exactly $\theta_x L(y) = M(x,y)$. This completes the proof. □

4. Projective dimension of indecomposable tilting modules

Now we are ready to prove the main result of this paper (see [Ma2, Conjecture 15(a)]).

Theorem 17. *Let $w \in W$. Then the projective dimension of the module $T(w)$ equals $\mathbf{a}(w)$.*

Theorem 17 follows from Lemmata 18 and 19 below. Both here and in the next section we use the technique for computation of extensions using complexes of tilting modules, developed in [MO1].

Lemma 18. *The projective dimension of $T(w)$ is at most $\mathbf{a}(w)$.*

Proof. Let $y \in W$. We start with the following computation (here the notation $\mathcal{O}$ means that we consider ungraded versions of all modules):

$$\begin{array}{rcl} \mathrm{Ext}^i_{\mathcal{O}}(T(w), L(y)) & \cong & \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(T(w), L(y)[\![i]\!]) \\ & \cong & \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\theta_{w_0w}T(w_0), L(y)[\![i]\!]) \\ \text{(by adjunction)} & \cong & \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(T(w_0), \theta_{(w_0w)^{-1}}L(y)[\![i]\!]) \\ & \cong & \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(T(w_0), M((w_0w)^{-1}, y)[\![i]\!]). \end{array}$$

By Theorem 16(a), the module $M((w_0w)^{-1}, y)$ can be represented in the derived category by the complex $M((w_0w)^{-1}, y)^\bullet$. Applying Theorem 16(b), we have $\Phi M((w_0w)^{-1}, y)^\bullet \cong M(y^{-1}w_0, w)$. Hence, by Proposition 1(c), the complex $M((w_0w)^{-1}, y)^\bullet$ is concentrated in positions between $-\mathbf{a}(w)$ and $\mathbf{a}(w)$.

Moreover, the complex $M((w_0w)^{-1}, y)^\bullet$ consists of tilting modules, and the module $T(w_0)$ is a tilting module as well. Hence, by [Ha, Chapter III(2),Lemma 2.1], the space $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(T(w_0), M((w_0w)^{-1}, y)^\bullet[\![i]\!])$ can be computed already in the homotopy category. However, if $i > \mathbf{a}(w)$, then from the previous paragraph it follows that all nonzero components of the complex $M((w_0w)^{-1}, y)^\bullet[\![i]\!]$ are in negative positions. As $T(w_0)$ is in position zero, we obtain that the morphism space from $T(w_0)$ to $M((w_0w)^{-1}, y)^\bullet[\![i]\!]$ in the homotopy category is zero. This implies $\mathrm{Ext}^i_{\mathcal{O}}(T(w), L(y)) = 0$ for all $i > \mathbf{a}(w)$ and all $x \in W$ and the claim of the lemma follows. □

Lemma 19. *The projective dimension of $T(w)$ is at least $\mathbf{a}(w)$.*

Proof. Let $\mathbf{R}$ and $\mathbf{L}$ denote the right and the left cells of w, respectively, and $\mathbf{T} \in \{\mathbf{R}, \mathbf{L}\}$. Recall (see for example the detailed explanation in [Ka, Section 5]) that

$$\mathbf{a}(w) = \min\{i \in \mathbb{Z} : \exists x \in \mathbf{T} \text{ s.t. } \mathrm{Hom}_{\mathtt{A}}(P(x)\langle -i\rangle, P(e)) \neq 0\}$$

(such x will be the Duflo involution in $\mathbf{T}$). Using the Ringel self-duality of $\mathcal{O}_0$ (which accounts to the application of T_{w_0} followed by $\star$) and using [AS] we obtain

$$\mathbf{a}(w) = \min\{i \in \mathbb{Z} : \exists x \in \mathbf{T} \text{ s.t. } \mathrm{Hom}_{\mathtt{A}}(T(w_0)\langle -i\rangle, T(w_0x)) \neq 0\}. \quad (3)$$

Consider modules $M(y^{-1}w_0, w)$, where $y \in W$. If y is such that $y^{-1}w_0$ runs through $\mathbf{R}$, then from Corollary 9 it follows that the socles of the modules $M(y^{-1}w_0, w)$ are $L(x)\langle -\mathbf{a}(w)\rangle$, $x \in \mathbf{R}$.

Applying Φ^{-1} and using Theorem 16, for y as above we obtain $\mathcal{M}(w^{-1}w_0, y)^i = 0$, $i > \mathbf{a}(w)$, while

$$\mathcal{M}(w^{-1}w_0, y)^{\mathbf{a}(w)} \cong T(w_0x^{-1}w_0)\langle \mathbf{a}(w)\rangle,$$

where $x \in \mathbf{R}$ ([Ma1, Theorem 3.3]). By (3) we can choose y such that $y^{-1}w_0 \in \mathbf{R}$ and for the corresponding x we have

$$\mathrm{Hom}_{\mathtt{A}}(T(w_0)\langle -\mathbf{a}(x^{-1}w_0)\rangle, T(w_0x^{-1}w_0)) \neq 0. \quad (4)$$

At the same time all tilting summands of $\mathcal{M}(w^{-1}w_0, y)^{\mathbf{a}(w)-1}$ have (up to shift) the form $T(z)$, where $z \leq_{LR} w_0x^{-1}w_0$. As $\mathbf{a}$ respects the two-sided order, we thus get

$$\mathrm{Hom}_{\mathtt{A}}(T(w_0)\langle -\mathbf{a}(x^{-1}w_0) + 1\rangle, T(z)) = 0 \quad (5)$$

for any such summand. From (4) it follows that there is a nonzero morphism from $T(w_0)\langle \mathbf{a}(w) - \mathbf{a}(x^{-1}w_0)\rangle$ to $\mathcal{M}(w^{-1}w_0, y)^{\bullet}[\![\mathbf{a}(w)]\!]$ in the category of complexes. From (5) it follows that there are no homotopy from $T(w_0)\langle \mathbf{a}(w) - \mathbf{a}(x^{-1}w_0)\rangle$ to $\mathcal{M}(w^{-1}w_0, y)^{\bullet}[\![\mathbf{a}(w)]\!]$. Hence there is a nonzero homomorphism from $T(w_0)\langle \mathbf{a}(w) - \mathbf{a}(x^{-1}w_0)\rangle$ to $\mathcal{M}(w^{-1}w_0, y)^{\bullet}[\![\mathbf{a}(w)]\!]$ in the homotopy category. Using [Ha, Chapter III(2),Lemma 2.1] and the adjunction, we thus get

$$\mathrm{Ext}_{\mathcal{O}}^{\mathbf{a}(w)}(T(w_0), M(w^{-1}w_0, y)) \cong \mathrm{Ext}_{\mathcal{O}}^{\mathbf{a}(w)}(T(w), L(y)) \neq 0.$$

The claim of the lemma follows. $\square$

5. Projective dimension of indecomposable injective modules

In this section we prove the second main result of this paper (see [Ma2, Conjecture 15(b)]).

Theorem 20. *Let $w \in W$. Then the projective dimension of the module $I(w)$ equals $2\mathbf{a}(w_0w)$.*

Proof. We prove this theorem by a descending induction with respect to the two-sided order on W. Note that the projective dimension of $I(w)$ is an invariant of a two-sided cell by [Ma2, Theorem 11]. If $w = w_0$, the module $I(w_0)$ is projective and thus has projective dimension zero, which agrees with our claim.

Fix $w \in W$, $w \neq w_0$, and assume that the claim of the theorem is true for all $x \in W$ such that $x >_{LR} w$.

Lemma 21. *The projective dimension of $I(w)$ is at most $2\mathbf{a}(w_0 w)$.*

Proof. Let d be the Duflo involution in the right cell of w. As projective dimension of $I(w)$ is an invariant of a two-sided cell by [Ma2, Theorem 11], it is enough to prove the claim in the case $w = d$. In this proof we consider all modules as ungraded.

Consider the injective module $\theta_d\theta_d I(e)$. We have

$$\mathrm{Hom}_{\mathcal{O}}(L(d), \theta_d\theta_d I(e)) \cong \mathrm{Hom}_{\mathcal{O}}(\theta_d L(d), \theta_d I(e)). \tag{6}$$

By Corollaries 3 and 8 we have that $\theta_d L(d)$ has simple socle $L(d)$ and hence the homomorphism space from $\theta_d L(d)$ to $\theta_d I(e) = I(d)$ is nonzero. From (6) it thus follows that $I(d)$ is a direct summand of $\theta_d\theta_d I(e)$. Hence to prove the statement of the lemma it is enough to show that the projective dimension of $\theta_d\theta_d I(e)$ is at most $2\mathbf{a}(w_0 w)$.

For all $i \in \{0, 1, \dots\}$ and all $y \in W$ by adjointness we have

$$\mathrm{Ext}^i_{\mathcal{O}}(\theta_d\theta_d I(e), L(y)) \cong \mathrm{Ext}^i_{\mathcal{O}}(\theta_d I(e), \theta_d L(y)).$$

The module $\theta_d L(y) = M(d, y)$ is represented in the derived category by the complex $\mathcal{M}(d, y)^\bullet$. From Theorem 16 and Proposition 1 it follows that this complex is concentrated between positions $-\mathbf{a}(w_0 d)$ and $\mathbf{a}(w_0 d)$. To proceed we need the following generalization of this observation:

Lemma 22. *For any $X \in \mathcal{O}$ the module $\theta_d X$ is represented in the derived category by some complex of tilting modules, concentrated between positions $-\mathbf{a}(w_0 d)$ and $\mathbf{a}(w_0 d)$.*

Proof. We prove the claim by induction on the length of X. For simple X the claim follows from the last paragraph before this lemma. To prove the induction step we consider a short exact sequence

$$0 \to Y \to X \to Z \to 0$$

such that Z is simple. Applying θ_d we get a short exact sequence

$$0 \to \theta_d Y \to \theta_d X \to \theta_d Z \to 0. \tag{7}$$

If $\theta_d Z = 0$, then $\theta_d X = \theta_d Y$ and the claim follows from the inductive assumption. Otherwise, from the inductive assumption we have a complex $\mathcal{C}^\bullet$ of tilting modules, concentrated between positions $-\mathbf{a}(w_0 d)$ and $\mathbf{a}(w_0 d)$, which represents $\theta_d Y$. As Z is simple, from the basis of the induction we have a complex $\mathcal{B}^\bullet$ of tilting modules, concentrated

between positions $-\mathbf{a}(w_0d)$ and $\mathbf{a}(w_0d)$, which represents $\theta_d Z$. The extension given by (7) corresponds to some morphism from $\mathcal{B}^\bullet[\![-1]\!]$ to $\mathcal{C}^\bullet$ in the homotopy category ([Ha, Chapter III(2),Lemma 2.1]). Taking the cone of this morphism we get a complex of tilting modules, concentrated between positions $-\mathbf{a}(w_0d)$ and $\mathbf{a}(w_0d)$, which represents $\theta_d X$. This completes the proof. □

By Lemma 22, we have that $\theta_d I(e)$ is represented in the derived category by some complex of tilting modules, concentrated between positions $-\mathbf{a}(w_0d)$ and $\mathbf{a}(w_0d)$.

Now for all $i \in \{0, 1, \dots\}$ we have

$$\mathrm{Ext}^i_{\mathcal{O}}(\theta_d I(e), \theta_d L(y)) \cong \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\theta_d I(e), \theta_d L(y)[\![i]\!]).$$

If we represent both $\theta_d I(e)$ and $\theta_d L(y)$ by the corresponding complexes of tilting modules, then the latter morphism space can be computed already in the homotopy category ([Ha, Chapter III(2),Lemma 2.1]). However, since both complexes are concentrated between positions $-\mathbf{a}(w_0d)$ and $\mathbf{a}(w_0d)$, it follows that for $i > 2\mathbf{a}(w_0d)$ the corresponding space in the homotopy category is zero. The claim of the lemma follows. □

Lemma 23. *The projective dimension of $I(w)$ is at least $2\mathbf{a}(w_0w)$.*

Proof. We again prove the statement in the case $w = d$ (the Duflo involution) and work with ungraded modules. Let X denote the cokernel of the natural inclusion $L(d) \hookrightarrow I(d) \cong \theta_d I(e)$. Consider the short exact sequence

$$0 \to \theta_d L(d) \to \theta_d\theta_d I(e) \to \theta_d X \to 0. \tag{8}$$

From the proof of Lemma 21 we know that the module $\theta_d\theta_d I(e)$ contains $I(d)$ as a direct summand. From [Ma2, (1)] it follows that the module $\theta_d\theta_d I(e)$ is a direct sum of injective modules $I(x)$, where $x \geq_{LR} d$. If $x >_L d$, then from the inductive assumption we know that the projective dimension of $I(x)$ is strictly less that $2\mathbf{a}(w_0w)$. Hence to prove the statement of the lemma it is enough to show that the projective dimension of $\theta_d\theta_d I(e)$ is at least $2\mathbf{a}(w_0w)$. To prove this it is enough to show that

$$\mathrm{Ext}^{2\mathbf{a}(w_0w)}_{\mathcal{O}}(\theta_d\theta_d I(e), \theta_d L(d)) \neq 0. \tag{9}$$

The module $\theta_d L(d)$ is self-dual by Proposition 1(a). It is also represented in the derived category by the complex $\mathcal{M}(d,d)^\bullet$, which is a linear complex of tilting modules and hence does not have trivial direct summands. By Theorem 16 and Proposition 1 we know that the leftmost nonzero position in $\mathcal{M}(d,d)^\bullet$ is $-\mathbf{a}(w_0d)$. Hence, by [MO1, Lemma 6 and Corollary 1] we have

$$\mathrm{Ext}^{2\mathbf{a}(w_0w)}_{\mathcal{O}}(\theta_d L(d), \theta_d L(d)) \neq 0. \tag{10}$$

Using Lemma 22 one shows that

$$\mathrm{Ext}^i_{\mathcal{O}}(\theta_d X, \theta_d L(d)) = 0$$

for all $i > 2\mathbf{a}(w_0 w)$. Hence, applying $\mathrm{Hom}_{\mathcal{O}}(\_, \theta_d L(d))$ to the short exact sequence (8) and going to the long exact sequence in homology we obtain a surjection

$$\mathrm{Ext}^{2\mathbf{a}(w_0 w)}_{\mathcal{O}}(\theta_d\theta_d I(e), \theta_d L(d)) \twoheadrightarrow \mathrm{Ext}^{2\mathbf{a}(w_0 w)}_{\mathcal{O}}(\theta_d L(d), \theta_d L(d)).$$

Now (9) follows from (10) and completes the proof. □

□

Above in the paper we have seen that the modules $M(x,y)$, $x, y \in W$, play important role in the combinatorics of the category $\mathcal{O}$. From this point of view the following problem looks rather natural:

Problem 24. Determine the projective dimension of $M(x,y)$ for all $x, y \in W$.

If $x = e$, then $M(x,y)$ is the simple module $L(y)$ and has projective dimension $2l(w_0) - l(w)$ ([Ma2, Proposition 6]). If $y = w_0$, then $M(x,y)$ is the tilting module $T(w_0 x)$ and has projective dimension $\mathbf{a}(w_0 x)$ (Theorem 17). In the general case I do not have any conjectural formula for the projective dimension of $M(x,y)$.

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